

On strong bounds of rate of convergence for regenerative processes

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Abstract

We give strong bounds for the rate of convergence of the regenerative process distribution to the stationary distribution in the total variation metric. These bounds are obtained by using coupling method. We propose this method for obtaining such bounds for the queueing regenerative processes.

Keywords *Regenerative process, queueing theory, rate of convergence, total variation metrics, coupling method*

1 Introduction

We study the rate of convergence of distribution of regenerative process to the stationary distribution in the total variation metric.

Many queueing processes are regenerative, and establishing bounds for the rate of their convergence is a very important problem for the practical applications of the queueing theory. Recall the definition of regenerative process.

Definition 1. The process $(X_t, t \geq 0)$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with a measurable state space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is regenerative, if there exists an increasing sequence $\{\theta_n\}$ ($n \in \mathbb{Z}_+$) of Markov moments with respect to the filtration $\mathcal{F}_{t \geq 0}$ such that the sequence

$$\{\Theta_n\} = \{X_{t+\theta_{n-1}} - X_{\theta_{n-1}}, \theta_n - \theta_{n-1}, t \in [\theta_{n-1}, \theta_n)\}, \quad n \in \mathbb{N}$$

consists of independent identically distributed (i.i.d.) random elements on $(\Omega, \mathcal{F}, \mathbf{P})$. If $\theta_0 \neq 0$, then the process $(X_t, t \geq 0)$ is called delayed.

Denote $\zeta_n \stackrel{\text{def}}{=} \theta_n - \theta_{n-1}$, and let $F(s) = \mathbf{P}\{\zeta_n \leq s\} = \mathbf{P}\{\zeta_1 \leq s\}$ ($n \in \mathbb{N}$) be the distribution function of regeneration period; we suppose that the distribution F is not lattice.

Also denote $\zeta_0 \stackrel{\text{def}}{=} \theta_0$, $\mathbb{F}(s) \stackrel{\text{def}}{=} \mathbf{P}\{\zeta_0 \leq s\}$.

Denote $\mathcal{P}_t^{X_0}(A) = \mathbf{P}\{X_t \in A\}$ for the process $(X_t, t \geq 0)$ with the initial state X_0 .

If $\mathbf{E} \zeta_i < \infty$, then for all X_0 we have $\mathcal{P}_t^{X_0} \implies \mathcal{P}$ where \mathcal{P} is the stationary distribution of the process $(X_t, t \geq 0)$.

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In the 70s A.A. Borovkov showed that if the condition

$$\mathbf{E} e^{\alpha \zeta} < \infty, \quad \alpha > 0$$

is satisfied, then for all set $A \in \mathcal{B}(\mathcal{X})$ and for all $a < \alpha$ there exists $\mathfrak{R}(a, A)$ such that for all $t > 0$:

$$|\mathbf{P}\{X_t \in A\} - \mathcal{P}(A)| < \frac{\mathfrak{R}(a, A)}{e^{at}},$$

and if the condition

$$\mathbf{E} \zeta^K < \infty, \quad k > 1$$

is satisfied, then for all set $A \in \mathcal{B}(\mathcal{X})$ and for all $k < K - 1$ there exists $R(k, A)$ such that for all $t > 0$:

$$|\mathbf{P}\{X_t \in A\} - \mathcal{P}(A)| < \frac{R(k, A)}{t^k},$$

where \mathcal{P} is the stationary distribution of $(X_t, t \geq 0)$, but bounds of the constants $\mathfrak{R}(\cdot)$ and $R(\cdot)$ are unknown.

Later this results was repeated and extended by probabilistic approaches, namely, by *modified* coupling method: shift-coupling and distributional shift-coupling [13, 14], ε -coupling [2, 7].

So, now it is known that if $\mathbf{E} \zeta_n^K < \infty$ for some $K > 1$ then for every $k < K - 1$ and $X_0 \in \mathcal{X}$

$$\lim_{t \rightarrow \infty} t^k \|\mathcal{P}_t^{X_0} - \mathcal{P}\|_{TV} = 0,$$

and if $\mathbf{E} e^{\alpha \zeta_i} < \infty$ for some $\alpha > 0$ then for every $a < \alpha$ and $X_0 \in \mathcal{X}$

$$\lim_{t \rightarrow \infty} e^{at} \|\mathcal{P}_t^{X_0} - \mathcal{P}\|_{TV} = 0$$

(see, e.g., [4, 2, 7, 9, 12, 13, 14] et al.).

So, we know two propositions:

Proposition 1. *If $\mathbf{E} \zeta_n^K < \infty$ for some $K > 1$, then for some $k < K - 1$ and for all $X_0 \in \mathcal{X}$ there exists $C(X_0, k)$ such that*

$$\|\mathcal{P}_t^{X_0} - \mathcal{P}\|_{TV} \leq \frac{C(X_0, k)}{(1+t)^k}.$$

Proposition 2. *If $\mathbf{E} e^{\alpha \zeta_i} < \infty$ for some $\alpha > 1$, then for all $a < \alpha$ and $X_0 \in \mathcal{X}$ there exists $\mathfrak{C}(X_0, \alpha)$ such that*

$$\|\mathcal{P}_t^{X_0} - \mathcal{P}\|_{TV} \leq \frac{\mathfrak{C}(X_0, a)}{e^{at}}.$$

Our goal is to find the bounds of the constants $C(X_0, k)$ and $\mathfrak{C}(X_0, a)$ in sufficiently wide conditions; note that the behavior of the constants $C(X_0, k)$ and $\mathfrak{C}(X_0, a)$ has been studied in [10, 11, 15, 16, 17, 18] for some special cases of regenerative processes.

In the sequel, we suppose that

$$\boxed{\int_{\{s: \exists F'(s)\}} F'(s) \, ds > 0; \quad \mathbf{E} \zeta_i \stackrel{\text{def}}{=} \mu_i < \infty} \quad (*)$$

2 Notations and the main results

Notation 1. The times θ_i (Definition 1) form the renewal process $N_t \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} \mathbf{1}(\theta_i \leq t)$. Denote $B_t \stackrel{\text{def}}{=} t - \theta_{N_t-1}$ (with $\theta_{-1} \stackrel{\text{def}}{=} 0$). The process $(B_t, t \geq 0)$ is the backward renewal time of the process N_t . \triangleright

Remark 1. The process $(B_t, t \geq 0)$ is the Markov regenerative process. \triangleright

Notation 2. For nondecreasing function $F(s)$ we put $F^{-1}(y) \stackrel{\text{def}}{=} \inf\{x : F(x) \geq y\}$. \triangleright

Notation 3. Denote

$$\tilde{F}(s) \stackrel{\text{def}}{=} \frac{\int_0^s (1 - F(u)) \, du}{\mu},$$

where

$$\mu \stackrel{\text{def}}{=} \int_0^{\infty} u \, dF(u) = \mathbf{E} \zeta.$$

Notation 4. Denote $F_a(s) \stackrel{\text{def}}{=} \frac{F(s+a) - F(a)}{1 - F(a)}$; $\mu_0 \stackrel{\text{def}}{=} \mathbf{E} \zeta_0$. \triangleright

Notation 5. Let $\mathcal{U}, \mathcal{U}', \mathcal{U}'', \mathcal{U}''', \mathcal{U}_i, \mathcal{U}_i', \mathcal{U}_i'', \mathcal{U}_i'''$ be independent uniformly distributed on $[0, 1)$ random variables on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$. \triangleright

Notation 6. Denote

$$\varphi(s) \stackrel{\text{def}}{=} \mathbf{1}(\exists F'(s)) \times (F'(s) \wedge \tilde{F}'(s))$$

and

$$\Phi(s) \stackrel{\text{def}}{=} \int_0^s \varphi(u) \, du;$$

the condition $(*)$ implies $\varkappa \stackrel{\text{def}}{=} \int_0^{\infty} \varphi(s) \, ds = \Phi(+\infty) > 0$. \triangleright

Notation 7. Denote

$$\Psi(s) \stackrel{\text{def}}{=} F(s) - \Phi(s), \quad \tilde{\Psi}(s) \stackrel{\text{def}}{=} \tilde{F}(s) - \Phi(s);$$

then $\Psi(+\infty) = \tilde{\Psi}(+\infty) = 1 - \varkappa$.

$$\text{Denote } \mathfrak{P}_a(\zeta_1) \stackrel{\text{def}}{=} \int_0^{\infty} e^{as} \, d\Psi(s).$$

Remark 2. Note that $\varkappa^{-1}\Phi(s)$ is the distribution function, and if $\varkappa < 1$ then $(1 - \varkappa)^{-1}\Psi(s)$ and $(1 - \varkappa)^{-1}\tilde{\Psi}(s)$ are the distribution function. \triangleright

If $\varkappa = 1$, then $\Phi(s) \equiv F(s) \equiv \tilde{F}(s) = 1 - e^{-\lambda s}$, and we put $\Psi(s) \equiv \tilde{\Psi}(s) \equiv 0$; in this case we assume $(1 - \varkappa)^{-1}\Psi(s) \equiv (1 - \varkappa)^{-1}\tilde{\Psi}(s) \equiv 0$, and $\Psi^{-1}(u) = \tilde{\Psi}^{-1}(u) = 0$. \triangleright

Notation 8. Put for independent random variables $\mathcal{U}, \mathcal{U}', \mathcal{U}''$ uniformly distributed on $[0, 1)$

$$\Xi(\mathcal{U}, \mathcal{U}', \mathcal{U}'') \stackrel{\text{def}}{=} \mathbf{1}(\mathcal{U} < \varkappa) \Phi^{-1}(\varkappa \mathcal{U}') + \mathbf{1}(\mathcal{U} \geq \varkappa) \Psi^{-1}((1 - \varkappa) \mathcal{U}'');$$

$$\tilde{\Xi}(\mathcal{U}, \mathcal{U}', \mathcal{U}'') \stackrel{\text{def}}{=} \mathbf{1}(\mathcal{U} < \varkappa) \Phi^{-1}(\varkappa \mathcal{U}') + \mathbf{1}(\mathcal{U} \geq \varkappa) \tilde{\Psi}^{-1}((1 - \varkappa) \mathcal{U}'').$$

▷

Remark 3. It is easy to see that

$$F(s) = \varkappa (\varkappa^{-1} \Phi(s)) + (1 - \varkappa) ((1 - \varkappa)^{-1} \Psi(s))$$

and

$$\tilde{F}(s) = \varkappa (\varkappa^{-1} \Phi(s)) + (1 - \varkappa) ((1 - \varkappa)^{-1} \tilde{\Psi}(s)).$$

Hence,

$$\mathbf{P}\{\Xi(\mathcal{U}, \mathcal{U}', \mathcal{U}'') \leq s\} = F(s), \quad \mathbf{P}\{\tilde{\Xi}(\mathcal{U}, \mathcal{U}', \mathcal{U}'') \leq s\} = \tilde{F}(s),$$

and

$$\mathbf{P}\{\Xi(\mathcal{U}, \mathcal{U}', \mathcal{U}'') = \tilde{\Xi}(\mathcal{U}, \mathcal{U}', \mathcal{U}'')\} = \varkappa.$$

▷

Notation 9. Denote

$$\begin{aligned} C(X_0, k) &\stackrel{\text{def}}{=} \mathbf{E} \zeta_0^k \varkappa \sum_{n=1}^{\infty} \left((n+1)^{k-1} (1 - \varkappa)^{n-1} \right) + \\ &\quad + \mathbf{E} \zeta_1^k \sum_{n=1}^{\infty} \left(\left(\varkappa n (n+2)^{k-1} + (n+1)^{k-1} \right) (1 - \varkappa)^{n-1} \right) \left[= \mathcal{C}(\zeta_0, k) \right]. \end{aligned}$$

▷

Notation 10. Denote $\mathfrak{C}(X_0, a) \stackrel{\text{def}}{=} \frac{\mathfrak{P}_a(\zeta_1) \mathbf{E} e^{a\zeta_0}}{1 - \mathfrak{P}_a(\zeta_1)} \left[= \mathcal{C}(\zeta_0, k) \right].$

▷

Theorem 1.

1. If $\mathbf{E}(\zeta_i)^K < \infty$ for some $K \geq 1$, then for all $k \in [1, K]$

$$\|\mathcal{P}_t^{X_0} - \mathcal{P}\|_{TV} \leq 2 \frac{C(X_0, k)}{t^\alpha}.$$

2. If $\mathbf{E} e^{\alpha \zeta_i} < \infty$, then there exists $a > 0$ such that $\mathfrak{P}_a < 1$, and

$$\|\mathcal{P}_t^{X_0} - \mathcal{P}\|_{TV} \leq 2 \frac{\mathfrak{C}(X_0, a)}{e^{at}}.$$

3 Idea of the proof of the Theorem 1.

3.1 Coupling method (see [8]).

To prove the Theorem, we will use the *modified coupling method*. The coupling method invented by W. Doeblin in [3] is used to obtain the bounds of the rate of convergence of a Markov process to the stationary regime.

Let $(X'_t, t \geq 0)$ and $(X''_t, t \geq 0)$ be two versions of Markov process $(X_t, t \geq 0)$ with different initial states x'_0 and x''_0 , correspondingly, and put

$$\mathcal{P}_t^{X'_0}(A) \stackrel{\text{def}}{=} \mathbf{P}\{X'_t \in A\}, \quad \mathcal{P}_t^{X''_0}(A) \stackrel{\text{def}}{=} \mathbf{P}\{X''_t \in A\},$$

and

$$\tau(X'_0, X''_0) \stackrel{\text{def}}{=} \inf\{t > 0 : X'_t = X''_t\}.$$

Let us know that $\mathcal{P}_t^{X_0} \Rightarrow \mathcal{P}$ for every initial state X_0 .

We suppose that $\mathbf{E} \varphi(\tau(X'_0, X''_0)) = C(X'_0, X''_0) < \infty$ for some positive increasing function $\varphi(t)$. Then

$$\left| \mathcal{P}_t^{X'_0}(A) - \mathcal{P}_t^{X''_0}(A) \right| \leq \mathbf{P}\{\tau(X'_0, X''_0) > t\} = \mathbf{P}\{\varphi(\tau(X'_0, X''_0)) > \varphi(t)\} \leq \frac{\mathbf{E} \varphi(\tau(X'_0, X''_0))}{\varphi(t)}$$

by the coupling inequality and Markov inequality.

By integration of this inequality with respect to the stationary measure \mathcal{P} we have

$$\left| \mathcal{P}_t^{X'_0}(A) - \mathcal{P}(A) \right| \leq (\varphi(t))^{-1} \int_{\mathcal{X}} \varphi(\tau(X'_0, X''_0)) d\mathcal{P}(X''_0) = \frac{\widehat{C}(X'_0)}{\varphi(t)}, \quad (1)$$

and

$$\left\| \mathcal{P}_t^{X'_0} - \mathcal{P} \right\|_{TV} \leq 2 \frac{\widehat{C}(X'_0)}{\varphi(t)}.$$

Emphasize that the application of the coupling method is possible only for the Markov processes. However, in queuing theory, usually the regenerative queueing processes are not Markov. Therefore, the state space of considered regenerative process must be extended so that the regenerative process with this state space would become Markov.

So, for the use of the coupling method for the arbitrary regenerative process $(X_t, t \geq 0)$ we must extend the state space \mathcal{X} of this process by such a way that the extended process $(\overline{X}_t, t \geq 0)$ with this extended state space $\overline{\mathcal{X}}$ is Markov. For markovization of non-Markov regenerative process we can include to the state $X_t, t \in [\theta_{n-1}, \theta_n)$ full history of the process on the time interval $[\theta_{n-1}, t]$: the extended process with the states $\overline{X}_t \stackrel{\text{def}}{=} \{X_s, s \in [\theta_{n-1}, t] | t < \theta_n\}$ is Markov and regenerative with the extended state space $\overline{\mathcal{X}}$.

Denote $\overline{\mathcal{P}}_t^{\overline{X}_0}(A) \stackrel{\text{def}}{=} \mathbf{P}\{\overline{X}_t \in A\}$ for the process $(\overline{X}_t, t \geq 0)$ with initial state \overline{X}_0 and $A \in \mathcal{B}(\overline{\mathcal{X}})$.

If $\mathbf{E} \zeta_i < \infty$ then $\overline{\mathcal{P}}_t^{\overline{X}_0} \Rightarrow \overline{\mathcal{P}}$.

If we can prove that $\left\| \overline{\mathcal{P}}_t^{\overline{X}_0} - \overline{\mathcal{P}} \right\|_{TV} \leq \varphi(t, \overline{X}_0)$ for all $t \geq 0$, then this inequality is true for the original non-Markov regenerative process $(X_t, t \geq 0)$.

For simplicity, we assume that the process $(\bar{X}_t, t \geq 0)$ is homogeneous Markov process, i.e. the transition function of this process in the period $[0, \theta_0]$ is the same as in the periods $[\theta_i, \theta_{i+1}]$, $i \geq 1$.

Thus, in the sequel we suppose that the regenerative process X_t is homogeneous Markov process.

Notice, that in general case $\mathbf{P}\{\tau(X'_0, X''_0) < \infty\} < 1$ (for the Markov processes in continuous time), and the “direct” use of coupling method is impossible.

3.1.1 Successful coupling (see [5]).

So, we will construct (in a special probability space) the paired stochastic process $\mathcal{Z}_t = ((Z'_t, Z''_t), t \geq 0)$ such that:

1. For all $t \geq 0$ $X'_t \stackrel{\mathcal{D}}{=} Z'_t$ and $X''_t \stackrel{\mathcal{D}}{=} Z''_t$.
2. $\mathbf{E} \tau(Z'_0, Z''_0) < \infty$, where $\tau(Z'_0, Z''_0) = \tau(\mathcal{Z}_0) \stackrel{\text{def}}{=} \inf\{t \geq 0 : Z'_t = Z''_t\}$.
3. $Z'_t = Z''_t$ for all $t \geq \tau(Z'_0, Z''_0)$.

The paired stochastic process $(\mathcal{Z}_t, t \geq 0) = ((Z'_t, Z''_t), t \geq 0)$ satisfying the conditions 1–3 is called *successful coupling*.

Remark 4. The processes $(Z'_t, t \geq 0)$ and $(Z''_t, t \geq 0)$ can be non-Markov, and its finite-dimensional distributions may differ from the finite-dimensional distributions of the processes $(X'_t, t \geq 0)$ and $(X''_t, t \geq 0)$ respectively. Furthermore, the processes $(Z'_t, t \geq 0)$ and $(Z''_t, t \geq 0)$ may be dependent. \triangleright

For all $A \in \mathcal{B}(\mathcal{X})$ we can use the coupling inequality in the form:

$$\begin{aligned} \left| \mathcal{P}_t^{X'_0}(A) - \mathcal{P}_t^{X''_0}(A) \right| &= |\mathbf{P}\{X'_t \in A\} - \mathbf{P}\{X''_t \in A\}| = \\ &= |\mathbf{P}\{Z'_t \in A\} - \mathbf{P}\{Z''_t \in A\}| \leq \mathbf{P}\{\tau(Z'_0, Z''_0) \geq t\} \leq \\ &\leq \frac{\mathbf{E} \varphi(\tau(Z'_0, Z''_0))}{\varphi(t)} \leq \frac{C(Z'_0, Z''_0)}{\varphi(t)} \quad (2) \end{aligned}$$

for some constant $C(Z'_0, Z''_0)$. As $Z_0^{(i)} = X_0^{(i)}$, the right-hand side of the inequality depends only on $X_0^{(i)}$. Then we can integrate the inequality (2) with respect to the measure \mathcal{P} as in (1):

$$\left| \mathcal{P}_t^{X'_0}(A) - \mathcal{P}(A) \right| \leq \frac{\int C(Z'_0, Z''_0) \mathcal{P}(dZ''_0)}{\varphi(t)} = \frac{\widehat{C}(Z'_0)}{\varphi(t)}. \quad (3)$$

The inequality (3) implies the bounds for $\left\| \mathcal{P}_t^{X'_0} - \mathcal{P} \right\|_{TV}$.

However, the integration in (3) gives some trouble.

3.2 Stationary modification of the coupling method.

We will construct a successful coupling $(\mathcal{Z}_t, t \geq 0) = (Z_t, \tilde{Z}_t)$ for the process $(X_t, t \geq 0)$ and its stationary version $(\tilde{X}_t, t \geq 0)$, then we will estimate the random variable

$$\tau(X_0) = \tau(Z_0) \stackrel{\text{def}}{=} \inf \left\{ t > 0 : Z_t = \tilde{Z}_t \right\}.$$

Then

$$|\mathcal{P}_t^{X_0}(A) - \mathcal{P}(A)| \leq \mathbf{P}\{\tau(X_0) > t\} \leq \frac{\mathbf{E}\varphi(\tau(X_0))}{\varphi(t)}$$

for all $A \in \mathcal{B}(\mathcal{X})$ by Markov inequality, and

$$\|\mathcal{P}_t^{X_0} - \mathcal{P}\|_{TV} \stackrel{\text{def}}{=} 2 \sup_{A \in \mathcal{B}(\mathcal{X})} |\mathcal{P}_t^{X_0}(A) - \mathcal{P}(A)| \leq 2 \frac{\mathbf{E}\varphi(\tau(X_0))}{\varphi(t)}.$$

Here $\mathcal{P}_t^{X_0}$ is the distribution of process $(X_t, t \geq 0)$ with the initial state X_0 .

Thus, the first step of our proof is the proof of the Theorem 2; the Theorem 1 is the consequence of the proof of the Theorem 2.

4 Implementation of idea.

Theorem 2. *There exists a successful coupling $(\mathcal{Z}_t, t \geq 0) = ((Z_t, \tilde{Z}_t), t \geq 0)$ for the process $(X_t, t \geq 0)$ and its stationary version $(\tilde{X}_t, t \geq 0)$.*

Proof. The proof of the Theorem 2 consists of 4 steps.

1. Let us prove the Theorem 2 for the backward renewal process $(B_t, t \geq 0)$ (Notation 1). We will give the construction of the successful coupling for the process $(B_t, t \geq 0)$ and its stationary version $(\tilde{B}_t, t \geq 0)$ (on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ in which the random variables $\mathcal{U}_i, \mathcal{U}'_i, \mathcal{U}''_i, \mathcal{U}'''_i$ are defined – see Notation 5).

I.e. we will find the uniform bounds for the convergence

$$\mathbf{P}\{B_t \leq s\} \longrightarrow \tilde{F}(s)$$

as $t \longrightarrow \infty$.

Recall that the processes B_t and \tilde{B}_t can be constructed as follows (see Fig. 1):

$$\zeta_0 \stackrel{\text{def}}{=} \mathbb{F}^{-1}(\mathcal{U}_0), \text{ and } \zeta_i \stackrel{\text{def}}{=} F^{-1}(\mathcal{U}_i) \text{ for } i > 0; \theta_i \stackrel{\text{def}}{=} \sum_{j=0}^i \zeta_j; Z_t \stackrel{\text{def}}{=} t - \max\{\theta_i : \theta_i \leq t\} \stackrel{\mathcal{D}}{=} B_t.$$

$$\tilde{\theta}_0 = \tilde{\zeta}_0 \stackrel{\text{def}}{=} \tilde{F}^{-1}(\mathcal{U}'_1), \text{ and } \tilde{\zeta}_i \stackrel{\text{def}}{=} F^{-1}(\mathcal{U}'_i) \text{ for } i > 0; \tilde{\theta}_i \stackrel{\text{def}}{=} \sum_{j=0}^i \tilde{\zeta}_j; \tilde{Z}_0 \stackrel{\text{def}}{=} F_{\tilde{\theta}_0}^{-1}(\mathcal{U}''_1);$$

$$\tilde{Z}_t \stackrel{\text{def}}{=} \mathbf{1}(t < \tilde{\theta}_0) (t + \tilde{Z}_0) + \mathbf{1}(t \geq \tilde{\theta}_0) (t - \max\{\tilde{\theta}_n : \tilde{\theta}_n \leq t\}) \stackrel{\mathcal{D}}{=} \tilde{B}_t.$$

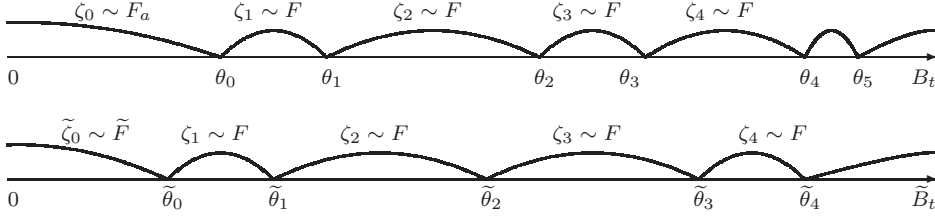


Figure 1: Construction of the independent processes $(B_t, t \geq 0)$ and $(\tilde{B}_t, t \geq 0)$

Remark 5. $\mathbf{P}\{\tilde{Z}_0 \leq s\} =$

$$= \int_0^\infty F_u(s) d\tilde{F}(u) = \int_0^\infty \frac{F(s+u) - F(u)}{1 - F(u)} \times \frac{1 - F(u)}{\mu} du = \frac{\int_0^s (1 - F(u)) du}{\mu} = \tilde{F}(s).$$

▷

This construction is the construction of independent version of the processes $(B_t, t \geq 0)$ and $(\tilde{B}_t, t \geq 0)$. Now we will transform this construction.

2. To construct a pair of (dependent) renewal processes, it is enough to construct all renewal times of both processes (times ϑ_i in the Fig.1). We will construct this pair $(\mathcal{Z}_t, t \geq 0) = ((Z_t, \tilde{Z}_t), t \geq 0)$ by induction.

Basis of induction.

Put

$$\theta_0 \stackrel{\text{def}}{=} \mathbb{F}^{-1}(\mathcal{U}_0), \quad \tilde{\theta}_0 \stackrel{\text{def}}{=} \tilde{F}^{-1}(\mathcal{U}'_0), \quad \tilde{Z}_0 \stackrel{\text{def}}{=} F_{\tilde{\theta}_0}^{-1}(\mathcal{U}''_0)$$

(recall that $\mathbf{P}\{\tilde{Z}_0 \leq s\} = \tilde{F}(s)$ – see Remark 5); and we put

$$Z_t \stackrel{\text{def}}{=} t, \quad \tilde{Z}_t \stackrel{\text{def}}{=} t + \tilde{Z}_0$$

for $t \in [0, \vartheta_0)$ where $\vartheta_0 \stackrel{\text{def}}{=} t_0 \wedge \tilde{t}_0$ (in the Fig.1 $\vartheta_0 = \tilde{\theta}_0$).

Inductive step.

Suppose that we have constructed the process $(\mathcal{Z}_t, t \geq 0)$ for $t \in [0, \vartheta_n)$, where $\vartheta_n = \theta_i \wedge \tilde{\theta}_j$.

There are three alternatives.

1. $\vartheta_n = \theta_i = \tilde{\theta}_j$ – in the Fig. 2 this situation occurs for the first time at the point ϑ_5 .

In this case we put

$$Z_{\vartheta_n} = \tilde{Z}_{\vartheta_n} = 0, \quad \theta_{i+1} = \tilde{\theta}_{j+1} = \vartheta_{n+1} = F^{-1}(\mathcal{U}_{n+1}) + \vartheta_n;$$

and $Z_t = \tilde{Z}_t \stackrel{\text{def}}{=} t - \vartheta_n$ for $t \in [\vartheta_n, \vartheta_{n+1})$. After the first coincidence (time $\tilde{\tau}$ in the Fig. 2) the processes $(Z_t, t \geq 0)$ and $(\tilde{Z}_t, t \geq 0)$ are identical.

2. $\vartheta_n = \tilde{\theta}_j < \theta_i$ (the times $\tilde{\theta}_0$ and $\tilde{\theta}_3$ in the Fig. 2).

In this case we put

$$\tilde{Z}_{\vartheta_n} = 0, \quad Z_{\vartheta_n} = Z_{\vartheta_n-0}, \quad \tilde{\theta}_{j+1} \stackrel{\text{def}}{=} \tilde{\theta}_j + F^{-1}(\mathcal{U}_{n+1});$$

and $\tilde{Z}_t \stackrel{\text{def}}{=} t - \vartheta_n$, $Z_t \stackrel{\text{def}}{=} t - \vartheta_n + Z_{\vartheta_n}$ for $t \in [\vartheta_n, \vartheta_{n+1})$ where $\vartheta_{n+1} \stackrel{\text{def}}{=} \theta_i \wedge \tilde{\theta}_{j+1}$.

3. $\vartheta_n = \theta_i < \tilde{\theta}_j$ (the times θ_0 , θ_1 and θ_2 in the Fig. 2).

In this case we put

$$\theta_{i+1} \stackrel{\text{def}}{=} \theta_i + \Xi(\mathcal{U}_{n+1}, \mathcal{U}'_{n+1}, \mathcal{U}''_{n+1}); \quad \tilde{\theta}_j \stackrel{\text{def}}{=} \theta_i + \tilde{A},$$

where $\tilde{A} = \tilde{\Xi}(\mathcal{U}_{n+1}, \mathcal{U}'_{n+1}, \mathcal{U}''_{n+1})$; and

$$Z_t \stackrel{\text{def}}{=} t - \vartheta_n, \quad \tilde{Z}_t \stackrel{\text{def}}{=} t - \vartheta_n + F_{\tilde{A}}^{-1}(\mathcal{U}'''_{n+1})$$

for $t \in [\vartheta_n, \vartheta_{n+1})$, where $\vartheta_{n+1} \stackrel{\text{def}}{=} \theta_i \wedge \tilde{\theta}_{j+1}$.

3. Let us prove that *the process* $(\mathcal{Z}_t, t \geq 0) = \left((Z_t, \tilde{Z}_t), t \geq 0 \right)$ *is a successful coupling for the processes* $(X_t, t \geq 0)$ and $(\tilde{X}_t, t \geq 0)$, and

$$\mathbf{E} \tau(X_0) \leq \mathbf{E} \tilde{\tau}(X_0) \leq \mathbf{E} \zeta_0 + 2\kappa^{-1} \mathbf{E} \zeta_1 < \infty.$$

Denote

$$\mathcal{E}_n \stackrel{\text{def}}{=} \{Z_{\theta_n} = \tilde{Z}_{\theta_n} | Z_{\theta_{n-1}} \neq \tilde{Z}_{\theta_{n-1}}\}; \quad \mathbf{P}(\mathcal{E}_n) = \kappa.$$

According to our construction of the pair $(\mathcal{Z}_t, t \geq 0)$, we have $\mathbf{P}\{Z_{\theta_0} \neq \tilde{Z}_{\theta_0}\} = 1$ because the distribution $\tilde{F}(s)$ is absolutely continuous.

Then

$$\mathbf{P}\{Z_{\theta_1} = \tilde{Z}_{\theta_1}\} = \mathbf{P}(\mathcal{E}_1) = \kappa, \quad \mathbf{P}\{Z_{\theta_{n+1}} = \tilde{Z}_{\theta_{n+1}} \& Z_{\theta_n} \neq \tilde{Z}_{\theta_n}\} = \mathbf{P}\left(\mathcal{E}_n \prod_{i=0}^{n-1} \overline{\mathcal{E}_i}\right),$$

and

$$\mathbf{P}\{\tilde{\tau} = \theta_{n+1}\} = \mathbf{P}(\mathcal{E}_n) \prod_{i=0}^{n-1} \mathbf{P}(\overline{\mathcal{E}_i}) = \kappa(1 - \kappa)^n.$$

Now, using the inequality

$$\mathbf{E}(\xi \times \mathbf{1}(\mathcal{E})) \mathbf{P}(\mathcal{E}) \leq \mathbf{E} \xi \tag{4}$$

for non-negative random variable ξ , and considering that

$$\mathbf{E}\left(\zeta_n \mathbf{1}\left(\bigcap_{i=1}^{n-1} \overline{\mathcal{E}_i}\right)\right) = \mathbf{E} \zeta_n$$

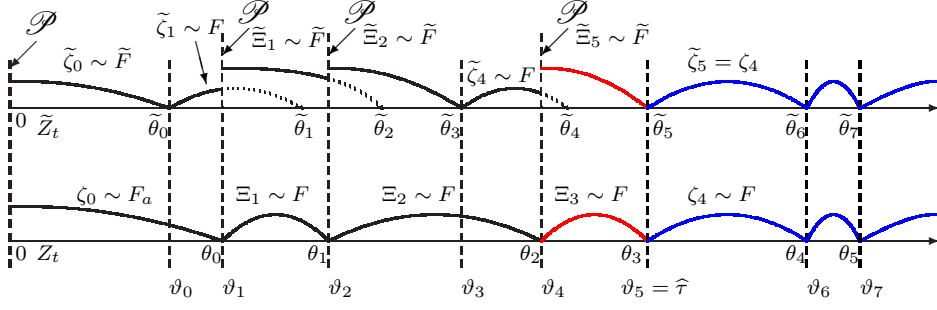


Figure 2: Construction of the successful coupling $(\mathcal{Z}_t, t \geq 0)$.

for $n > 0$, we have

$$\begin{aligned}
\mathbf{E}\tilde{\tau} &= \mathbf{E}\zeta_0 + \mathbf{E}(\zeta_1 \times \mathbf{1}(\mathcal{E}_1))\mathbf{P}(\mathcal{E}_1) + \mathbf{E}((\zeta_1 + \zeta_2) \times \mathbf{1}(\overline{\mathcal{E}}_1))\mathbf{1}(\mathcal{E}_2)\mathbf{P}(\overline{\mathcal{E}}_1\mathcal{E}_2) + \\
&+ \mathbf{E}((\zeta_1 + \zeta_2 + \zeta_3) \times \mathbf{1}(\overline{\mathcal{E}}_1)\mathbf{1}(\overline{\mathcal{E}}_2)\mathbf{1}(\mathcal{E}_3))\mathbf{P}(\overline{\mathcal{E}}_1\overline{\mathcal{E}}_2\mathcal{E}_3) + \dots + \\
&+ \mathbf{E}\left(\left(\sum_{i=1}^n \zeta_i\right) \times \mathbf{1}(\mathcal{E}_n) \prod_{i=1}^{n-1} \mathbf{1}(\overline{\mathcal{E}}_i)\right)\mathbf{P}\left(\mathcal{E}_n \prod_{i=1}^{n-1} \overline{\mathcal{E}}_i\right) + \dots \leq \\
&\leq \mathbf{E}\zeta_0 + \mathbf{E}\zeta_1 \left(1 + \kappa \sum_{i=0}^{\infty} (1 - \kappa)^i\right) + (1 - \kappa) \mathbf{E}\zeta_2 \left(1 + \kappa \sum_{i=0}^{\infty} (1 - \kappa)^i\right) + \\
&+ \dots + (1 - \kappa)^{n-1} \mathbf{E}\zeta_n \left(1 + \kappa \sum_{i=0}^{\infty} (1 - \kappa)^i\right) + \dots = \mathbf{E}\zeta_0 + 2\kappa^{-1}\mathbf{E}\zeta_1. \quad (5)
\end{aligned}$$

Theorem 2 is proved. \square

4. Now we return to the Markov regenerative process $(X_t, t \geq 0)$. The regeneration times $\theta_0, \theta_1, \dots$ of the process $(X_t, t \geq 0)$ form an embedded backward renewal process $(Y_t, t \geq 0)$; $Y_t \stackrel{\text{def}}{=} t - \max\{\theta_i : \theta_i \leq t\}$ (the backward renewal time of the embedded renewal process) with the distribution of the renewal time $\mathbf{P}\{\zeta \leq s\} = F(s)$; the length of i -th regeneration period is $\theta_i - \theta_{i-1} = \zeta_i \stackrel{\mathcal{D}}{=} \zeta$ ($i > 1$).

We will apply the coupling method for extended process $(\mathbb{X}_t, t \geq 0) = ((X_t, Y_t), t \geq 0)$ on the extended state space $\widehat{\mathcal{X}} = \mathcal{X} \times \mathbb{R}_+$.

The random element $\mathfrak{X}_i = \{X_t, t \in [\theta_{i-1}, \theta_i]\}$ depends on the random variable $\zeta_i = \theta_i - \theta_{i-1}$; for $A \in \mathcal{B}(\mathcal{X})$ denote

$$\mathcal{G}_a(s, A) \stackrel{\text{def}}{=} \mathbf{P}\{X_{\theta_{i-1}+s} \in A | \zeta_i = a\}, \quad s \in [0, a];$$

$\mathcal{G}_a(s, A)$ specify a conditional distribution of $(X_t, t \geq 0)$ on the time period $[\theta_{i-1}, \theta_i]$ given $\{\theta_i - \theta_{i-1} = a\}$.

Therefore if we know all regeneration times of the process $(X_t, t \geq 0)$, then we know the (conditional) distribution of the process $(X_t, t \geq 0)$ in every time after the first regeneration time

t_0 : this distribution is determined by the conditional distribution $\mathcal{G}_{\zeta_i}(s, A)$ of the random elements \mathfrak{X}_i .

Also the first regeneration time depends on the initial state; denote

$$\mathcal{H}_a(A) \stackrel{\text{def}}{=} \mathbf{P}\{X_0 \in A | \theta_0 = a\} = \mathbf{P}\{X_t \in A | (\min\{\theta_i : \theta_i \geq t\} - t) = a\};$$

$\mathcal{H}_a(A)$ specify a conditional distribution of $(X_t, t \geq 0)$ given $\{\zeta_t^* = a\}$, where $\zeta_t^* \stackrel{\text{def}}{=} \min\{t_i : t_i \geq t\} - t$ is a residual time of regeneration period at the time t .

Now we will construct the successful coupling for extended process $(\mathbb{X}_t, t \geq 0) = ((X_t, Y_t), t \geq 0)$ and its stationary version $(\tilde{\mathbb{X}}_t, t \geq 0) = ((\tilde{X}_t, \tilde{Y}_t), t \geq 0)$.

For this aim we construct the successful coupling $(\mathscr{W}, t \geq 0)_t = ((W_t, \widetilde{W}_t), t \geq 0)$ for the backward renewal process $(Y_t, t \geq 0)$ and its stationary version $(\tilde{Y}_t, t \geq 0)$ considering that the first renewal time θ_0 of the process $(Y_t, t \geq 0)$ has a distribution $\mathbb{F}(s)$.

After construction of renewal points $\{\theta_i\}$ of the process $(W_t, t \geq 0)$ and renewal points $\{\tilde{\theta}_i\}$ of the process $(\widetilde{W}_t, t \geq 0)$ we can complete them to the pairs $((Z_t, W_t), t \geq 0) \stackrel{\mathcal{D}}{=} ((X_t, Y_t), t \geq 0)$ and $((\tilde{Z}_t, \widetilde{W}_t), t \geq 0) \stackrel{\mathcal{D}}{=} ((\tilde{X}_t, \tilde{Y}_t), t \geq 0)$ by using $\mathcal{G}_a(s, A)$ and $\mathcal{H}_a(A)$.

In the construction of the processes $(W_t, t \geq 0)$ and $(\widetilde{W}_t, t \geq 0)$ we can apply the technics used in the proof of the Theorem 2 in such a way that

$$\tau(X_0) = \inf \left\{ t : W_t = \widetilde{W}_t \right\} \leq t_0 + \sum_{i=1}^{\nu} \zeta_i,$$

where $\mathbf{P}\{\nu > n\} = (1 - \varkappa)^n$.

So, for $t \geq \tau(X_0)$ we have $W_t = \widetilde{W}_t$, by construction of the backward renewal processes $(W_t, t \geq 0)$ and $(\widetilde{W}_t, t \geq 0)$.

Then for $t \geq \tau(X_0)$ we have

$$\mathcal{P}_t^{X_0}(A) = \mathbf{P}\{X_t \in A\} = \mathbf{P}\{\tilde{X}_t \in A\} = \mathcal{P}(A)$$

as the distribution of X_t and \tilde{X}_t (after the first renewal point) is determined only by renewal points.

Proof of the Theorem 1.

1. Using the inequality (4) and Jensen's inequality for $k \geq 1$ and $a_i \leq 0$ in the form

$$\left(\sum_{i=1}^n a_i \right)^k \leq n^{k-1} \sum_{i=1}^n a_i^k$$

we have from (5):

$$\begin{aligned}
\mathbf{E}(\tau(X_0))^k &\leq \mathbf{E}(\tilde{\tau}(X_0))^k \leq \\
&\leq \mathbf{E} \left(\mathbf{P} \left(\mathcal{E}_n \prod_{i=1}^{n-1} \overline{\mathcal{E}}_i \right) \sum_{n=1}^{\infty} \left((n+1)^{k-1} \left(\zeta_0^k + \sum_{i=1}^n \zeta_i^k \right) \mathbf{1}_{(\mathcal{E}_n)} \prod_{i=1}^{n-1} \mathbf{1}_{(\overline{\mathcal{E}}_i)} \right) \right) \leq \\
&\leq \mathbf{E} \zeta_0^k \varkappa \sum_{n=1}^{\infty} \left((n+1)^{k-1} (1-\varkappa)^{n-1} \right) + \mathbf{E} \zeta_1^k \sum_{n=1}^{\infty} \left(\left(\varkappa n (n+2)^{k-1} + (n+1)^{k-1} \right) (1-\varkappa)^{n-1} \right) = \\
&= C(X_0, k) \left[= \mathcal{C}(\zeta_0, k) \right],
\end{aligned}$$

this inequality implies the statement 1 of the Theorem 1.

2. Using the inequality (4) and considering that $\mathbf{E} e^{\alpha \zeta_1} \mathbf{1}_{(\overline{\mathcal{E}}_i)} = \mathfrak{P}_\alpha(\zeta_1)$ we have from (5):

$$\begin{aligned}
\mathbf{E} e^{\alpha \tau}(X_0) &\leq \mathbf{E} e^{\alpha \tilde{\tau}}(X_0) \leq \mathbf{E} \left(\mathbf{P} \left(\mathcal{E}_n \prod_{i=1}^{n-1} \overline{\mathcal{E}}_i \right) \times \sum_{n=1}^{\infty} \left(e^{\alpha \left(\zeta_0 + \sum_{i=1}^n \zeta_i \right)} \mathbf{1}_{(\mathcal{E}_n)} \prod_{i=1}^{n-1} \mathbf{1}_{(\overline{\mathcal{E}}_i)} \right) \right) \leq \\
&\leq \mathbf{E} e^{\alpha \zeta_0} \sum_{n=1}^{\infty} \left((1-\varkappa)^n \prod_{i=1}^n \mathbf{E} \left(e^{\alpha \zeta_1} \mathbf{1}_{(\overline{\mathcal{E}}_i)} \right) \right) = \mathbf{E} e^{\alpha \zeta_0} \sum_{n=1}^{\infty} \left((1-\varkappa)^n \left(\int_0^\infty e^{\alpha s} d \frac{\Psi(s)}{1-\varkappa} \right) \right) = \\
&= \mathbf{E} e^{\alpha \zeta_0} \sum_{n=1}^{\infty} (\mathfrak{P}_\alpha(\zeta_1))^n = \frac{\mathfrak{P}_\alpha(\zeta_1) \mathbf{E} e^{\alpha \zeta_0}}{1 - \mathfrak{P}_\alpha(\zeta_1)} = \mathfrak{C}(X_0, \alpha) \left[= \mathcal{C}(\zeta_0, \alpha) \right],
\end{aligned}$$

this inequality implies the statement 2 of the Theorem 1. \square

5 Applying to the queueing theory.

In the queueing theory the distribution of the regenerative process period is often unknown. But often the regeneration period can be split into two parts, usually this is a busy period and an idle period. And as a rule the idle period has a known non-discrete distribution. So, in this situation the queueing process has an embedded *alternating renewal process*.

If the bounds of moments of a busy period are also known, then we can apply our construction for embedded alternating renewal process by some modification.

This modification is a construction of a successful coupling for alternating renewal process $(X_t, t \geq 0)$ and its stationary version, namely.

Let $(X_t, t \geq 0)$ be an alternating renewal process having two states, 1 and 2, say. The time of the stay of the process in a state i has the distribution function $F_i(s) = \mathbf{P} \{ \zeta^{(i)} \leq s \}$, and periods of stay of the process $(X_t, t \geq 0)$ in states 1 and 2 alternate. This process is non-Markov.

We complement the state of the process by the time, during which the process located continually in this state: if the completed process is the process $(Y_t, t \geq 0) = ((n_t, x_t), t \geq 0)$ (denote $n(Y_t) = n_t, x(Y_t) = x_t$), then at the time t the process is in the state n_t , and $x_t \stackrel{\text{def}}{=} t - \sup\{s < t : n(Y_s) \neq n_t\}$ (for definiteness, we assume $\mathbb{F}(s) = F_a(s)$, $x_0 = a$, and $x_t = a + t$ for $t \in [0, \inf\{s > 0 : n_s \neq n_0\})$).

The Markov regenerative process $(Y_t, t \geq 0)$ has a state space $\{1, 2\} \times \mathbb{R}_+$.

Denote $c(n) \stackrel{\text{def}}{=} \mathbf{1}(n=2) + 2 \times \mathbf{1}(n=1)$: here $c(1) = 2, c(2) = 1$.

If $Y_0 = (i, a)$, then the process Y_t changes its first component $n(Y_t)$ at the times $\zeta^{(a,i)} = \theta_{0,i}, \theta_{0,c(i)}, \theta_{1,i}, \theta_{1,c(i)}, \dots$.

Denote

$$\zeta_j^{(i)} = \theta_{j,c(i)} - \theta_{j,i} \stackrel{\mathcal{D}}{=} \zeta^{(i)}; \quad \zeta_j^{(c(i))} = \theta_{j,i} - \theta_{j-1,c(i)} \stackrel{\mathcal{D}}{=} \zeta^{(c(i))};$$

$$\mathbf{P}\{\zeta^{(a,i)} \leq s\} = \frac{F_i(a+s) - F_i(s)}{1 - F_i(s)}.$$

We assume that the distribution function $F_1(s)$ satisfies the condition $(*)$, and random variables $\zeta^{(a,i)}, \zeta_j^{(1)}, \zeta_j^{(2)}$ are mutually independent.

Suppose that $\mathbf{E}\varphi(\zeta^{(i)}) < \infty$ for some increasing positive function $\varphi(t)$.

If $\mathbf{E}\zeta^{(i)} = \mu_i < \infty$, then distribution $\mathcal{P}_t^{Y_0}$ of the process Y_t with every initial state Y_0 weakly convergent to the stationary distribution \mathcal{P} ; for the stationary version \tilde{Y}_t of the process Y_t we have

$$\mathbf{P}\{n(\tilde{Y}_t) = 1\} = \frac{\mu_1}{\mu_1 + \mu_2} \stackrel{\text{def}}{=} p.$$

If we know only an estimate $\mu_2 \leq m_2$, then

$$p \geq \rho \stackrel{\text{def}}{=} \frac{\mu_1}{\mu_1 + m_2}.$$

For construction of successful coupling $(\mathcal{X}_t, t \geq 0) = ((Z_t, \tilde{Z}_t), t \geq 0)$ for the processes $(Y_t, t \geq 0)$ and $(\tilde{Y}_t, t \geq 0)$ we will again construct the times when at least one of them change the first component.

At the times t'_i such that $n(Y_{t'_i-0}) = 2$ and $n(Y_{t'_i+0}) = 1$ we use (with probability p) the random variables Ξ_i for Y_t and $\tilde{\Xi}_i$ for \tilde{Y}_t ;

$$\mathbf{P}\{\Xi_i \leq s\} = F_1(s), \quad \mathbf{P}\{\tilde{\Xi}_i \leq s\} = \tilde{F}_1(s) = \frac{\int_0^s (1 - F_1(u)) du}{\mu_1},$$

and

$$\mathbf{P}\{\Xi_i = \tilde{\Xi}_i\} = \varkappa = \int_{\{s: \exists F'_1(s)\}} F'_1(s) \wedge \tilde{F}'_1(s) ds.$$

And with probability $q = 1 - p$ at the time θ'_i we use a procedure of the prolongation of alternating renewal process \tilde{Y}_t by using the distribution $\tilde{F}_2(s) = (\mu_2)^{-1} \int_0^s (1 - F_2(u)) du$.

So,

$$\tilde{\tau}(Y_0) \stackrel{\text{def}}{=} \inf \left\{ t > 0 : Z_t = \tilde{Z}_t \right\} \leq \theta'_1 + \zeta_\nu^{(1)} + \sum_{i=1}^{\nu-1} \left(\zeta_i^{(1)} + \zeta_i^{(2)} \right),$$

where $\mathbf{P}\{\nu > n\} = (1 - p\kappa)^n \left[\leq (1 - \rho\kappa)^n \right]$.

Hence, if $\mathfrak{P}_a(\zeta_1) \mathbf{E} e^{a\zeta^{(1)}} < 1$, we can find a bound $\mathfrak{C}(Y_0, a)$ for $\mathbf{E} e^{a\tilde{\tau}(Y_0)}$:

$$\mathbf{E} e^{a\tilde{\tau}(Y_0)} \leq \mathfrak{C}(Y_0, a).$$

Therefore we have

$$\left\| \mathcal{P}_t^{Y_0} - \mathcal{P} \right\|_{TV} \leq 2 \frac{\mathfrak{C}(Y_0, a)}{e^{at}}.$$

Furthermore, if the distributions F_1 and F_2 satisfy the condition (*), then the inequality $\mathbf{E} a\zeta^{(i)} < \infty$ for some $a > 0$ implies an existence of $\alpha > 0$ such that $\mathfrak{P}_\alpha(\zeta^{(i)}) < 1$, and we can find the bounds $\mathfrak{C}(Y_0, a)$ for $\mathbf{E} a\tilde{\tau}(Y_0)$.

And if $\mathbf{E}(\zeta^{(i)})^K < \infty$ for some $K \geq 1$, then we can estimate $\mathbf{E}(\tilde{\tau}(Y_0))^k$ for $k \in [1, K]$:

$$\mathbf{E}(\tau(Y_0))^k \leq C(Y_0, k),$$

where the constant $C(Y_0, k)$ is calculated by the schema similar to the schema of the proof of the Theorem 1.

Again we have

$$\left\| \mathcal{P}_t^{Y_0} - \mathcal{P} \right\|_{TV} \leq 2 \frac{C(Y_0, k)}{t^k}.$$

If the regeneration period of queueing process $(Q_t, t \geq 0)$ can be split into two independent parts, then this process has an embedded alternating renewal process.

Firstly we will extend this queueing process $(Q_t, t \geq 0)$ to the Markov process $(X_t, t \geq 0)$.

Then we will complete the process $(X_t, t \geq 0)$ by the embedded alternating renewal process $(Y_t, t \geq 0)$ completed by the time from the last change of its state (as in previous part).

Using the technique of the proof of the Theorem 1 we can find the bounds for the convergence rate for extended queueing process; also this bounds are useful for the original process $(Q_t, t \geq 0)$.

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